

Quantum curve and 4D limit of melting crystal model

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Abstract

This paper considers the problems of quantum spectral curves and 4D limit for the melting crystal model of 5D SUSY $U(1)$ Yang-Mills theory on $\mathbb{R}^4 \times S^1$. The partition function $Z(\mathbf{t})$ deformed by an infinite number of external potentials is a tau function of the KP hierarchy with respect to the coupling constants $\mathbf{t} = (t_1, t_2, \dots)$. A single-variate specialization $Z(x)$ of $Z(\mathbf{t})$ satisfies a q -difference equation representing the quantum spectral curve of the melting crystal model. In the limit as the radius R of S^1 in $\mathbb{R}^4 \times S^1$ tends to 0, it turns into a difference equation for a 4D counterpart $Z_{4D}(X)$ of $Z(x)$. This difference equation reproduces the quantum spectral curve of Gromov-Witten theory of \mathbb{CP}^1 . $Z_{4D}(X)$ is obtained from $Z(x)$ by letting $R \rightarrow 0$ under an R -dependent transformation $x = x(X, R)$ of x to X . A similar prescription of 4D limit can be formulated for $Z(\mathbf{t})$ with an R -dependent transformation $\mathbf{t} = \mathbf{t}(\mathbf{T}, R)$ of \mathbf{t} to $\mathbf{T} = (T_1, T_2, \dots)$. This yields a 4D counterpart $Z_{4D}(\mathbf{T})$ of $Z(\mathbf{t})$. $Z_{4D}(\mathbf{T})$ agrees with a generating function of all-genus Gromov-Witten invariants of \mathbb{CP}^1 . Fay-type bilinear equations for $Z_{4D}(\mathbf{T})$ can be derived from similar equations satisfied by $Z(\mathbf{t})$. The bilinear equations imply that $Z_{4D}(\mathbf{T})$, too, is a tau function of the KP hierarchy.

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1 Introduction

The melting crystal model [14] is a statistical model of 5D SUSY Yang-Mills theory on $\mathbb{R}^4 \times S^1$ [18] in the self-dual background [19, 20]. The partition function is a sum over all possible shapes (represented by plane partitions) of 3D Young diagrams. The name of the model originates in the physical interpretation of the complement of a 3D Young diagram in the positive octant of \mathbb{R}^3 as a melting crystal corner. By the method of diagonal slicing [24], the partition function can be converted to a sum over ordinary partitions. This sum reproduces the Nekrasov partition function of instantons in 5D SUSY Yang-Mills theory.

In the previous work [17], we studied the simplest case that amounts to $U(1)$ gauge theory. The main subject was an integrable structure of the partition function deformed by an infinite number of external potentials. The deformed partition function $Z(\mathbf{t}, s)$ depends on the coupling constants $\mathbf{t} = (t_1, t_2, \dots)$ of those potentials and a discrete variable $s \in \mathbb{Z}$. We proved, with the aid of symmetries of a quantum torus algebra, that $Z(\mathbf{t}, s)$ is essentially a tau function of the 1D Toda hierarchy [35]. This result has been extended to some other types of melting crystal models [30, 31].

An open problem raised therein is to find an appropriate prescription for the 4D limit as the radius R of S^1 in $\mathbb{R}^4 \times S^1$ tends to 0. The melting crystal model of $U(1)$ gauge theory has two parameters q, Q . By setting these parameters in a particular R -dependent form and letting $R \rightarrow 0$, the undeformed partition function $Z = Z(\mathbf{0}, 0)$ converges to the 4D Nekrasov function Z_{4D} [19, 20]. It is not so straightforward to achieve the 4D limit of the deformed partition function $Z(\mathbf{t}, s)$. In a naive prescription [17], all coupling constants other than t_1 decouple from $Z(\mathbf{t}, s)$ in the limit as $R \rightarrow 0$. On the other hand, a deformation $Z_{4D}(\mathbf{t}, s)$ of Z_{4D} by an infinite number of external potentials is proposed in the literature [13]. What we need is an R -dependent transformation $\mathbf{t} = \mathbf{t}(\mathbf{T}, R)$ of \mathbf{t} to a new set of coupling constants $\mathbf{T} = (T_1, T_2, \dots)$ such that $Z(\mathbf{t}(\mathbf{T}, R), s)$ converges to $Z_{4D}(\mathbf{T}, s)$ as $R \rightarrow 0$. This is a problem that we address in this paper.

Another problem tackled here is to derive the so called quantum spectral curves. This problem is inspired by the work of Dunin-Barkowski et al. [5] on Gromov-Witten theory of \mathbb{CP}^1 . They derived a quantum spectral curve of \mathbb{CP}^1 in the perspective of topological recursion [6, 7, 21]. Since the deformed 4D Nekrasov function $Z_{4D}(\mathbf{T}, s)$ of $U(1)$ gauge theory coincides with a generating function of all genus Gromov-Witten invariants of \mathbb{CP}^1 [11, 23], it will be natural to reconsider this issue from the point of view of the melting crystal model. Recently, we proposed a new approach to quantum mirror curves in topological string theory [32]. This approach is based on

the notions of Kac-Schwarz operators [10, 28] and generating operators [2, 3] in the KP hierarchy [26, 27]. $Z(\mathbf{t}, s)$ may be thought of as a set of KP tau functions labelled by $s \in \mathbb{Z}$. In particular, $Z(\mathbf{t}) = Z(\mathbf{t}, 0)$ resembles the tau functions in topological string theory. Our method developed for topological string theory can be applied to $Z(\mathbf{t})$ to derive a quantum spectral curve. This quantum curve is represented by a q -difference equation for a single-variate specialization $Z(x)$ of $Z(\mathbf{t})$. As $R \rightarrow 0$, this equation turns into a difference equation that was derived by Dunin-Barkowski et al. as the quantum spectral curve of \mathbb{CP}^1 .

Let us stress that these two problems are closely related. To derive the 4D limit of the quantum spectral curve, we choose an R -dependent transformation $x = x(X, R)$ to a new variable X such that $Z(x(X, R))$ converges to a function $Z_{4D}(X)$ as $R \rightarrow 0$. It is this function $Z_{4D}(X)$ that was considered by Dunin-Barkowski et al. [5] and shown to satisfy the aforementioned difference equation. Moreover, $Z_{4D}(X)$ turns out to be a single-variate specialization of a multi-variate function $Z_{4D}(\mathbf{T})$ that is obtained as the limit of $Z(\mathbf{t}(\mathbf{T}, R))$ as $R \rightarrow 0$ in the sense explained above. $Z_{4D}(\mathbf{T})$ is also a deformation of Z_{4D} by an infinite number of external potentials. It is straightforward to extend $Z_{4D}(\mathbf{T})$ to a function $Z_{4D}(\mathbf{T}, s)$ that depends on $s \in \mathbb{Z}$.

As a byproduct of this prescription of 4D limit, we shall show that $Z_{4D}(\mathbf{T})$ satisfies a set of Fay-type bilinear equations. These bilinear equations are known to characterize tau functions of the KP hierarchy [1, 26, 33]. This implies that $Z_{4D}(\mathbf{T})$, too, is a tau function of the KP hierarchy. By a similar characterization of tau functions of the Toda hierarchy [29, 34], one can deduce that $Z_{4D}(\mathbf{T}, s)$ is a tau function of the 1D Toda hierarchy. Let us recall once again that $Z_{4D}(\mathbf{T}, s)$ is a generating function of all-genus Gromov-Witten invariants of \mathbb{CP}^1 [11, 23]. The well known fact, referred to as the Toda conjecture, on Gromov-Witten theory of \mathbb{CP}^1 [4, 8, 15, 25] can be thus explained in a different perspective.

This paper is organized as follows. Section 2 is a brief review of the melting crystal model. Combinatorial and fermionic expressions of the deformed partition function $Z(\mathbf{t})$ are introduced. The fermionic expression is further converted to a form that fits into the method of our work on quantum mirror curves of topological string theory [32]. Section 3 presents the quantum spectral curve of the melting crystal model. The single-variate specialization $Z(x)$ of $Z(\mathbf{t})$ is introduced, and shown to satisfy a q -difference equation representing the quantum spectral curve. The computations are mostly parallel to the case of topological string theory. Section 4 deals with the issue of 4D limit. The R -dependent transformations $x = x(X, R)$ and $\mathbf{t} = \mathbf{t}(\mathbf{T}, R)$ are introduced, and $Z(x(X, R))$ and $Z(\mathbf{t}(\mathbf{T}, R))$ are shown to converge as $R \rightarrow 0$. The functions $Z_{4D}(X)$ and $Z_{4D}(\mathbf{T})$ obtained in this limit

are computed explicitly. The difference equation for $Z_{4D}(X)$ is derived, and confirmed to agree with the result of Dunin-Barkowski et al. [5]. Section 5 is devoted to Fay-type bilinear equations. A three-term bilinear equation plays a central role here. The bilinear equation for $Z(\mathbf{t})$ is shown to turn into a similar bilinear equation for $Z_{4D}(\mathbf{T})$ as $R \rightarrow 0$. The detail of consideration on $Z(\mathbf{t}, s)$ and $Z_{4D}(\mathbf{T}, s)$ is omitted. Section 6 concludes this paper.

2 Melting crystal model

2.1 Partition function of 3D Young diagrams

The partition function of the simplest melting crystal model with a single parameter q is the sum

$$Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|} \quad (2.1)$$

of the Boltzmann weight $q^{|\pi|}$ over the set \mathcal{PP} of all plane partitions. The plane partition $\pi = (\pi_{ij})_{i,j=1}^{\infty}$ represent a 3D Young diagram that consists of stacks of unit cubes of height π_{ij} put on the unit squares $[i-1, i] \times [j-1, j]$ of the plane. $|\pi|$ denotes the volume of the 3D Young diagram:

$$|\pi| = \sum_{i,j=1}^{\infty} \pi_{ij}.$$

By the method of diagonal slicing [24], one can convert the sum (2.1) over \mathcal{PP} to a sum over the set \mathcal{P} of all ordinary partitions $\lambda = (\lambda_i)_{i=1}^{\infty}$ as

$$Z = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(q^{-\rho})^2, \quad (2.2)$$

where $s_{\lambda}(q^{-\rho})$ is the special value (a kind of principal specialization) of the infinite-variate Schur function $s_{\lambda}(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \dots)$, at

$$\mathbf{x} = q^{-\rho} = (q^{1/2}, q^{3/2}, \dots, q^{i-1/2}, \dots).$$

Moreover, by the Cauchy identities of Schur functions [12], one can rewrite the sum (2.2) into an infinite product:

$$Z = \prod_{i,j=1}^{\infty} (1 - q^{i+j-1})^{-1} = \prod_{n=1}^{\infty} (1 - q^n)^{-n}.$$

This infinite product is known as the MacMahon function.

The special value $s_\lambda(q^{-\rho})$ has the hook-length formula [12]

$$s_\lambda(q^{-\rho}) = \frac{q^{-\kappa(\lambda)/4}}{\prod_{(i,j) \in \lambda} (q^{-h(i,j)/2} - q^{h(i,j)/2})}, \quad (2.3)$$

where $\kappa(\lambda)$ is the commonly used notation

$$\kappa(\lambda) = 2 \sum_{(i,j) \in \lambda} (j - i) = \sum_{i=1}^{\infty} ((\lambda_i - i + 1/2)^2 - (-i + 1/2)^2),$$

and $h(i, j)$ denote the hook length

$$h(i, j) = (\lambda_i - j) + ({}^t\lambda_j - i) + 1$$

of the cell (i, j) in the Young diagram of shape λ . ${}^t\lambda_i$'s are the parts of the conjugate partition ${}^t\lambda$ that represents the transposed Young diagram. Thus $s_\lambda(q^{-\rho})$ is a q -deformation of the number

$$\frac{\dim \lambda}{|\lambda|!} = \frac{1}{\prod_{(i,j) \in \lambda} h(i,j)}, \quad (2.4)$$

where $|\lambda| = \sum_{i=1}^{\infty} \lambda_i$, and $\dim \lambda$ is the dimension of the irreducible representation of the symmetric group S_d , $d = |\lambda|$. The square of (2.4) is called the Plancherel measure on the symmetric group, and plays a central role in Gromov-Witten/Hurwitz theory of \mathbb{CP}^1 as well [22, 23, 25].

2.2 Deformation by external potentials

We now introduce a parameter Q and an infinite set of coupling constants $\mathbf{t} = (t_1, t_2, \dots)$, and deform (2.2) as

$$Z(\mathbf{t}) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} e^{\phi(\mathbf{t}, \lambda)}, \quad (2.5)$$

where

$$\phi(\mathbf{t}, \lambda) = \sum_{k=1}^{\infty} t_k \phi_k(\lambda).$$

The external potentials $\phi_k(\lambda)$ are defined as

$$\phi_k(\lambda) = \sum_{i=1}^{\infty} (q^{k(\lambda_i - i + 1)} - q^{k(-i + 1)}). \quad (2.6)$$

The sum on the right hand side of (2.6) is a finite sum because only a finite number of λ_i 's are non-zero. These potentials are q -analogues of the so called Casimir invariants of the infinite symmetric group S_∞ , which we shall encounter in the 4D limit.

The following fact is a consequence of our previous work on the melting crystal model [17]. We shall refine this statement in the subsequent consideration.

Theorem 1. *$Z(\mathbf{t})$ is a tau function of the KP hierarchy with time variables $\mathbf{t} = (t_1, t_2, \dots)$.*

Actually, $Z(\mathbf{t})$ is a member of a set of functions $Z(\mathbf{t}, s)$, $s \in \mathbb{Z}$, considered in our previous work:

$$Z(\mathbf{t}, s) = \sum_{\lambda \in \mathcal{P}} s_\lambda (q^{-\rho})^2 Q^{|\lambda|+s(s+1)/2} e^{\phi(\mathbf{t}, s, \lambda)}, \quad (2.7)$$

where

$$\begin{aligned} \phi(\mathbf{t}, s, \lambda) &= \sum_{k=1}^{\infty} t_k \phi_k(s, \lambda), \\ \phi_k(s, \lambda) &= \sum_{i=1}^{\infty} (q^{k(\lambda_i - i + 1 + s)} - q^{k(-i + 1 + s)}) + \frac{1 - q^{ks}}{1 - q^k} q^k. \end{aligned}$$

As shown therein, $Z(\mathbf{t}, s)$ is, up to a simple multiplier, a tau function of the 1D Toda hierarchy, hence a collection of tau functions of the KP hierarchy labelled by s [35]. Consequently, $Z(\mathbf{t}) = Z(\mathbf{t}, 0)$ is a KP tau function.

The relation to the 1D Toda hierarchy is explained in the fermionic formalism of integrable hierarchies [9, 16]. The fermionic formalism plays a central role in our derivation of quantum spectral curves as well.

2.3 Fermionic expression of partition function

Let ψ_n, ψ_n^* , $n \in \mathbb{Z}$, be the creation-annihilation operators¹ of 2D charged free fermion theory with the anti-commutation relations

$$\psi_m \psi_n^* + \psi_n^* \psi_m = \delta_{m+n, 0}, \quad \psi_m \psi_n + \psi_n \psi_m = 0, \quad \psi_m^* \psi_n^* + \psi_n^* \psi_m^* = 0,$$

¹For the sake of convenience, as in our previous work [17], we label these operators with integers rather than half-integers. The free fermion fields are defined as $\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}$ and $\psi^*(z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n}$.

and $|0\rangle, \langle 0|$, $s \in \mathbb{Z}$, the vacuum vectors of the fermionic Fock and dual Fock spaces that satisfy the vacuum conditions

$$\begin{aligned}\psi_n^*|0\rangle &= 0 \quad \text{for } n > 0, & \psi_n|0\rangle &= 0 \quad \text{for } n \geq 0, \\ \langle 0|\psi_n &= 0 \quad \text{for } n < 0, & \langle 0|\psi_n^* &= 0 \quad \text{for } n \leq 0.\end{aligned}$$

The charge-0 sectors of the Fock spaces are spanned by the excited states $|\lambda\rangle, \langle\lambda|$, $\lambda \in \mathcal{P}$:

$$\begin{aligned}|\lambda\rangle &= \psi_{-\lambda_1}\psi_{-\lambda_2+1}\cdots\psi_{-\lambda_n+n-1}\psi_{-n+1}^*\cdots\psi_{-1}^*\psi_0^*|0\rangle, \\ \langle\lambda| &= \langle 0|\psi_0\psi_1\cdots\psi_{n-1}\psi_{\lambda_n-n+1}^*\cdots\psi_{\lambda_2-1}^*\psi_{\lambda_1}^*,\end{aligned}$$

where n is chosen so that $\lambda_i = 0$ for $i > n$. In particular, $|\emptyset\rangle$ and $\langle\emptyset|$ agree with the vacuum states. The charge- s sector of the Fock spaces are spanned by similar vectors $|s, \lambda\rangle, \langle s, \lambda|$, $\lambda \in \mathcal{P}$.

The fermionic expression of the aforementioned partition functions employs the normally ordered fermion bilinears

$$\begin{aligned}L_0 &= \sum_{n \in \mathbb{Z}} n : \psi_{-n} \psi_n^* :, & K &= \sum_{n \in \mathbb{Z}} (n - 1/2)^2 : \psi_{-n} \psi_n^* :, \\ H_k &= \sum_{n \in \mathbb{Z}} q^{kn} : \psi_{-n} \psi_n^* :, & J_k &= \sum_{n \in \mathbb{Z}} : \psi_{k-n} \psi_n^* :, \\ & & : \psi_m \psi_n^* : &= \psi_m \psi_n^* - \langle 0 | \psi_m \psi_n^* | 0 \rangle,\end{aligned}$$

the vertex operators [24, 36]

$$\Gamma_{\pm k}(x) = \exp\left(\sum_{k=1}^{\infty} \frac{x^k}{k} J_{\pm k}\right), \quad \Gamma'_{\pm k}(x) = \exp\left(-\sum_{k=1}^{\infty} \frac{(-x)^k}{k} J_{\pm k}\right),$$

and their multi-variate extensions

$$\Gamma_{\pm k}(x_1, x_2, \dots) = \prod_{i \geq 1} \Gamma_{\pm k}(x_i), \quad \Gamma'_{\pm k}(x_1, x_2, \dots) = \prod_{i \geq 1} \Gamma'_{\pm k}(x_i).$$

The action of these operators on the fermionic Fock space leaves the charge-0 sector invariant. L_0 , K and H_k are diagonal with respect to $|\lambda\rangle$'s:

$$\langle\lambda|L_0|\mu\rangle = |\lambda|\delta_{\lambda\mu}, \quad \langle\lambda|K|\mu\rangle = \kappa(\lambda)\delta_{\lambda\mu}, \quad \langle\lambda|H_k|\mu\rangle = \phi_k(\lambda)\delta_{\lambda\mu}. \quad (2.8)$$

The matrix elements of the vertex operators are the skew Schur functions $s_{\lambda/\mu}(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \dots)$:

$$\langle\lambda|\Gamma_{-}(\mathbf{x})|\mu\rangle = \langle\mu|\Gamma_{+}(\mathbf{x})|\lambda\rangle = s_{\lambda/\mu}(\mathbf{x}), \quad (2.9)$$

$$\langle\lambda|\Gamma'_{-}(\mathbf{x})|\mu\rangle = \langle\mu|\Gamma'_{+}(\mathbf{x})|\lambda\rangle = s_{\iota_{\lambda}/\iota_{\mu}}(\mathbf{x}). \quad (2.10)$$

One can use these building blocks to rewrite the combinatorial definition (2.5) of $Z(\mathbf{t})$ as

$$Z(\mathbf{t}) = \langle 0 | \Gamma_+(q^{-\rho}) Q^{L_0} e^{H(\mathbf{t})} \Gamma_-(q^{-\rho}) | 0 \rangle, \quad (2.11)$$

where

$$H(\mathbf{t}) = \sum_{k=1}^{\infty} t_k H_k.$$

As shown in our previous work with the aid of symmetries of a quantum torus algebra [17], (2.11) can be converted to the following form. This implies that $Z(\mathbf{t})$ is a tau function of the KP hierarchy.

Theorem 2.

$$Z(\mathbf{t}) = \exp \left(\sum_{k=1}^{\infty} \frac{q^k t_k}{1 - q^k} \right) \langle 0 | \exp \left(\sum_{k=1}^{\infty} (-1)^k q^{k/2} t_k J_k \right) g_1 | 0 \rangle, \quad (2.12)$$

where

$$g_1 = q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{K/2}. \quad (2.13)$$

Remark 1. We could have removed the rightmost two factors $\Gamma_+(q^{-\rho}) q^{K/2}$ from g_1 because they leave the vacuum vector $|0\rangle$ invariant:

$$\Gamma_+(q^{-\rho}) q^{K/2} |0\rangle = |0\rangle. \quad (2.14)$$

g_1 is defined as shown in (2.13) to enjoy the algebraic relations

$$J_k g_1 = g_1 J_k \quad \text{for } k = 1, 2, \dots,$$

which play the role of a reduction condition from the 2D Toda hierarchy to the 1D Toda hierarchy [17].

One can rewrite (2.12) further to clarify its characteristic as a tau function of the KP hierarchy. First of all, the exponential prefactor on the right hand side can be taken inside the vev as

$$Z(\mathbf{t}) = \langle 0 | \exp \left(\sum_{k=1}^{\infty} (-1)^k q^{k/2} t_k J_k \right) g_2 | 0 \rangle, \quad (2.15)$$

where

$$g_2 = \exp \left(\sum_{k=1}^{\infty} \frac{(-1)^k q^{k/2}}{1 - q^k} J_{-k} \right) g_1.$$

This is a consequence of the commutation relation

$$[J_m, J_n] = m\delta_{m+n,0}$$

among J_n 's that span the $U(1)$ current (or Heisenberg) algebra. Remarkably, the operator generated in front of g_1 , too, is related to a vertex operator as

$$\exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k q^{k/2}}{1-q^k} J_{-k}\right) = \Gamma'_-(q^{-\rho})^{-1}.$$

Thus g_2 can be expressed as

$$g_2 = \Gamma'_-(q^{-\rho})^{-1} q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{K/2}. \quad (2.16)$$

Moreover, the multiplier $(-1)^k q^{k/2}$ of t_k in (2.15) can be removed by the scaling relation

$$\sum_{k=1}^{\infty} (-1)^k q^{k/2} t_k J_k = (-q^{1/2})^{-L_0} \cdot \sum_{k=1}^{\infty} t_k J_k \cdot (-q^{1/2})^{L_0}.$$

(2.15) thereby turns into the more standard expression

$$Z(\mathbf{t}) = \langle 0 | \exp\left(\sum_{k=1}^{\infty} t_k J_k\right) (-q^{1/2})^{L_0} g_2 | 0 \rangle$$

as a tau function of the KP hierarchy [9, 16].

Remark 2. In the previous work [17], we used the operator

$$W_0 = \sum_{n \in \mathbb{Z}} n^2 : \psi_{-n} \psi_n^* :$$

in place of K . Accordingly, the fermionic expression of the partition functions presented therein takes a slightly different form. This does not affect the essential part of the fermionic expression.

3 Quantum spectral curve of melting crystal model

3.1 Single-variate specialization

Let $Z(x)$ denote the single-variate specialization of $Z(\mathbf{t})$ obtained by substituting

$$t_k = -\frac{q^{-k/2} x^k}{k}, \quad k = 1, 2, \dots \quad (3.1)$$

The combinatorial definition (2.5) of $Z(\mathbf{t})$ and its fermionic expressions (2.12) and (2.15) are accordingly specialized as follows.

Lemma 1.

$$Z(x) = \sum_{\lambda \in \mathcal{P}} s_{\lambda} (q^{-\rho})^2 Q^{|\lambda|} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2} x}{1 - q^{-i + 1/2} x}. \quad (3.2)$$

Proof. Substituting (3.1) for $\phi(\mathbf{t}, \lambda)$ yields

$$\begin{aligned} \phi(\mathbf{t}, \lambda) &= - \sum_{i,k=1}^{\infty} \frac{q^{-k/2} x^k}{k} (q^{k(\lambda_i - i + 1)} - q^{k(-i + 1)}) \\ &= \sum_{i=1}^{\infty} (\log(1 - q^{\lambda_i - i + 1/2} x) - \log(1 - q^{-i + 1/2} x)), \end{aligned}$$

hence

$$e^{\phi(\mathbf{t}, \lambda)} = \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2} x}{1 - q^{-i + 1/2} x}.$$

□

Lemma 2.

$$Z(x) = \prod_{i=1}^{\infty} (1 - q^{i-1/2} x) \cdot \langle 0 | \Gamma'_+(x) g_1 | 0 \rangle = \langle 0 | \Gamma'_+(x) g_2 | 0 \rangle. \quad (3.3)$$

Proof. Substituting (3.1) in (2.12) and (2.15) yields

$$\exp \left(\sum_{k=1}^{\infty} \frac{q^k t_k}{1 - q^k} \right) = \exp \left(- \sum_{i,k=1}^{\infty} q^{k(i-1/2)} x^k \right) = \prod_{i=1}^{\infty} (1 - q^{i-1/2} x)$$

(cf. the computation in the proof of the previous lemma) and

$$\exp \left(\sum_{k=1}^{\infty} (-1)^k q^{k/2} t_k J_k \right) = \exp \left(- \sum_{k=1}^{\infty} \frac{(-x)^k}{k} J_k \right) = \Gamma'_+(x).$$

□

As we shall see in the next section, the combinatorial expression (3.2) of $Z(x)$ has a desirable form from which one can derive the equation of quantum curve of Dunin-Barkowski et al. [5]. To apply the method of our previous work [32], however, it is more convenient to have $\Gamma_+(x)$ rather than $\Gamma'_+(x)$ in the fermionic expression (3.3) of $Z(x)$. This problem can be resolved by the following transformation rule of matrix elements of fermionic operators under conjugation of partitions [36]:

Lemma 3.

$$\begin{aligned}\langle \lambda | L_0 | \lambda \rangle &= \langle {}^t \lambda | L_0 | {}^t \lambda \rangle, & \langle \lambda | K | \lambda \rangle &= -\langle {}^t \lambda | K | {}^t \lambda \rangle, \\ \langle \lambda | \Gamma_{\pm}(\mathbf{x}) | \mu \rangle &= \langle {}^t \lambda | \Gamma'_{\pm}(\mathbf{x}) | {}^t \mu \rangle.\end{aligned}$$

Proof. These identities are consequences of (2.8), (2.9), (2.10) and the following property of $\kappa(\lambda)$:

$$\kappa({}^t \lambda) = -\kappa(\lambda).$$

□

We can apply this rule to $\Gamma'_+(x)$ and the building blocks of g_2 to rewrite (3.3) as

$$Z(x) = \langle 0 | \Gamma_+(x) g'_2 | 0 \rangle, \quad (3.4)$$

where

$$g'_2 = \Gamma_-(q^{-\rho})^{-1} q^{-K/2} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) Q^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) q^{-K/2}. \quad (3.5)$$

Remark 3. The existence of two expressions, (3.3) and (3.4), of $Z(x)$ carries over to $Z(\mathbf{t})$:

$$\begin{aligned}Z(\mathbf{t}) &= \langle 0 | \exp \left(\sum_{k=1}^{\infty} (-1)^k q^{k/2} t_k J_k \right) g_2 | 0 \rangle \\ &= \langle 0 | \exp \left(- \sum_{k=1}^{\infty} q^{k/2} t_k J_k \right) g'_2 | 0 \rangle.\end{aligned} \quad (3.6)$$

Note that the third formula of Lemma 3 implies the identity

$$\langle \mu | \exp \left(\sum_{k=1}^{\infty} (-1)^k q^{k/2} t_k J_k \right) | \lambda \rangle = \langle {}^t \mu | \exp \left(- \sum_{k=1}^{\infty} q^{k/2} t_k J_k \right) | {}^t \lambda \rangle$$

among the matrix elements of the exponential operators in (3.6).

Having obtained the fermionic expression (3.4) containing $\Gamma_+(x)$, we now remove the other Γ_+ 's from (3.4). This is the last step for applying the method of our previous work [32].

Lemma 4.

$$Z(x) = \prod_{n=1}^{\infty} (1 - Q q^n)^{-n} \cdot \langle 0 | \Gamma_+(x) g | 0 \rangle, \quad (3.7)$$

where

$$g = \Gamma_-(q^{-\rho})^{-1} q^{-K/2} \Gamma'_-(q^{-\rho}) \Gamma'_-(Q q^{-\rho}). \quad (3.8)$$

Proof. The rightmost two factors $\Gamma'_+(q^{-\rho})q^{-K/2}$ of (3.5), like the operators in (2.14), can be removed from (3.4). One can use the scaling relation

$$\Gamma'_\pm(x_1, x_2, \dots)Q^{L_0} = Q^k \Gamma'_\pm(Q^{\pm 1}x_1, Q^{\pm 1}x_2, \dots)$$

and the commutation relation

$$\Gamma'_+(x_1, x_2, \dots)\Gamma'_-(y_1, y_2, \dots) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} \cdot \Gamma'_-(y_1, y_2, \dots)\Gamma'_+(x_1, x_2, \dots)$$

of the vertex operators [24, 36] to rewrite the product of the three operators in the middle of (3.5) as

$$\begin{aligned} \Gamma'_+(q^{-\rho})Q^{L_0}\Gamma'_-(q^{-\rho}) &= Q^{L_0}\Gamma'_+(Qq^{-\rho})\Gamma'_-(q^{-\rho}) \\ &= \prod_{n=1}^{\infty} (1 - Qq^n)^{-n} \cdot Q^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(Qq^{-\rho}) \\ &= \prod_{n=1}^{\infty} (1 - Qq^n)^{-n} \cdot \Gamma'_-(Qq^{-\rho})Q^{L_0}\Gamma'_+(Qq^{-\rho}). \end{aligned}$$

The two factors $Q^{L_0}\Gamma'_+(Qq^{-\rho})$ in the last line hit the vacuum vector $|0\rangle$ and disappear. What remains are the constant $\prod_{n=1}^{\infty} (1 - Qq^n)^{-n}$ and the operator g . \square

Remark 4. This is actually a proof of the identity

$$g'_2|0\rangle = \prod_{n=1}^{\infty} (1 - Qq^n)^{-n} \cdot g|0\rangle \quad (3.9)$$

of vectors in the fermionic Fock space.

3.2 Generating operator of admissible basis

We now borrow the idea of generating operators from the work of Alexandrov et al. [2, 3]. A point of the Sato Grassmannian can be represented by a linear subspace W of the space $V = \mathbb{C}((x))$ of formal Laurent series [26, 27]. The generating operator is a linear automorphism G of V that maps $W_0 = \text{Span}\{x^{-j}\}_{j=0}^{\infty}$ to W , so that an admissible basis $\{\Phi_j(x)\}_{j=0}^{\infty}$ of W can be expressed as

$$\Phi_j(x) = Gx^{-j}. \quad (3.10)$$

In the fermionic formalism of the KP hierarchy [9, 16], W corresponds to a vector $|W\rangle$ of the fermionic Fock space. The associated tau function can be defined as

$$\tau(\mathbf{t}) = \langle 0 | \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) | W \rangle.$$

Its special value at

$$\mathbf{t} = [x] = (x, x^2/2, \dots, x^k/k, \dots) \quad (3.11)$$

is related to the first member $\Phi_0(x)$ of an admissible basis of W as

$$\tau([x]) = \langle 0 | \Gamma_+(x) | W \rangle = C \Phi_0(x), \quad (3.12)$$

where C is a nonzero constant.

If $|W\rangle$ is generated from the vacuum vector $|0\rangle$ by an operator g as

$$|W\rangle = g|0\rangle,$$

and g is a special operator, such as a product of vertex operators and particular diagonal operators, then one can find G rather easily from g by the correspondence

$$L_0 \longleftrightarrow D = x \frac{d}{dx}, \quad K \longleftrightarrow \left(D - \frac{1}{2} \right)^2, \quad J_k \longleftrightarrow x^{-k}, \quad \text{etc.}, \quad (3.13)$$

between fermion bilinears and differential operators. This is the way how Alexandrov et al. derived the generating operator for various types of Hurwitz numbers [3]. We did similar computations for tau functions in topological string theory [32].

One can interpret the last fermionic expression (3.7) of $Z(x)$ in the same sense. Namely, as shown in (3.12) in the general setting, $Z(x)$ may be thought of as the first member $\Phi_0(x)$ of an admissible basis of the subspace W determined by (3.8). One can see from (3.9) that the associated tau function

$$\tau(\mathbf{t}) = \langle 0 | \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) g | 0 \rangle$$

amounts to the second expression of (3.6). Since the time variables of $\tau(\mathbf{t})$ are rescaled as $t_k \rightarrow -q^{k/2} t_k$ therein, the specialization (3.1) of $Z(\mathbf{t})$ corresponds to the standard specialization (3.11) of $\tau(\mathbf{t})$.

It is now straightforward to find the generating operator G from (3.8). According to (3.13), $q^{-K/2}$ corresponds to a differential operator of infinite order:

$$q^{-K/2} \longleftrightarrow q^{-(D-1/2)^2/2}.$$

The three vertex operators correspond to multiplication operators:

$$\begin{aligned}\Gamma_-(q^{-\rho})^{-1} &\longleftrightarrow \exp\left(-\sum_{k=1}^{\infty} \frac{q^{k/2}x^k}{k}\right) = \prod_{i=1}^{\infty} (1 - q^{i-1/2}x), \\ \Gamma'_-(q^{-\rho}) &\longleftrightarrow \prod_{i=1}^{\infty} (1 + q^{i-1/2}x), \\ \Gamma'_-(Qq^{-\rho}) &\longleftrightarrow \prod_{i=1}^{\infty} (1 + Qq^{i-1/2}x).\end{aligned}$$

The generating operator is given by a product of these operators as follows.

Theorem 3. *The generating operator G for the subspace $W \subset V$ determined by (3.8) can be expressed as*

$$G = \prod_{i=1}^{\infty} (1 - q^{i-1/2}x) \cdot q^{-(D-1/2)^2/2} \cdot \prod_{i=1}^{\infty} (1 + q^{i-1/2}x)(1 + Qq^{i-1/2}x). \quad (3.14)$$

3.3 Derivation of quantum spectral curve

Since the structure of the generating operator (3.14) resembles those of tau functions in topological string theory [32], we define the Kac-Schwarz operator A in essentially the same form,² namely,

$$A = G \cdot q^D \cdot G^{-1}.$$

The members $\Phi_j(x)$ of the admissible basis (3.10) thereby satisfy the linear equations

$$A\Phi_j(x) = q^{-j}\Phi_j(x).$$

In particular, the equation

$$(A - 1)\Phi_0(x) = 0$$

for $\Phi_0(x)$ (equivalently, $\tau([x])$) represents the quantum spectral curve. As we show below, A is a q -difference operator of finite order.

Lemma 5.

$$\begin{aligned}A &= (1 + q^{1/2}xq^{-D}(1 - q^{1/2}x)^{-1}) (1 + Qq^{1/2}xq^{-D}(1 - q^{1/2}x)^{-1}) \\ &\quad \times (1 - q^{1/2}x)q^D.\end{aligned} \quad (3.15)$$

²This operator amount to the inverse A^{-1} of the Kac-Schwarz operator A considered therein.

Proof. One can compute $A = G \cdot q^D \cdot G^{-1}$ step by step. The first step is to apply the last infinite product of (3.14) and its inverse to q^D . This can be carried out with the aid of the operator identity

$$q^D \cdot x = qxq^D$$

as follows:

$$\begin{aligned} & \prod_{i=1}^{\infty} (1 + q^{i-1/2}x)(1 + Qq^{i-1/2}x) \cdot q^D \cdot \prod_{i=1}^{\infty} (1 + Qq^{i-1/2}x)^{-1}(1 + q^{i-1/2}x)^{-1} \\ &= \prod_{i=1}^{\infty} (1 + q^{i-1/2}x)(1 + Qq^{i-1/2}x) \cdot \prod_{i=1}^{\infty} (1 + Qq^{i+1/2}x)^{-1}(1 + q^{i+1/2}x)^{-1} \cdot q^D \\ &= (1 + q^{1/2}x)(1 + Qq^{1/2}x)q^D. \end{aligned}$$

The next step is to apply $q^{-(D-1/2)^2/2}$ and its inverse to the last operator. This can be achieved by the identity

$$q^{-(D-1/2)^2/2} \cdot x \cdot q^{(D-1/2)^2/2} = xq^{-D}$$

as follows:

$$\begin{aligned} & q^{-(D-1/2)^2/2} \cdot (1 + q^{1/2}x)(1 + Qq^{1/2}x)q^D \cdot q^{(D-1/2)^2/2} \\ &= (1 + q^{1/2}xq^{-D})(1 + Qq^{1/2}xq^{-D})q^D. \end{aligned}$$

The last step is to apply the first infinite product of (3.14) and its inverse to the last operator. Since q^{-D} and q^D are thereby transformed as

$$\begin{aligned} & \prod_{i=1}^{\infty} (1 - q^{i-1/2}x) \cdot q^{-D} \cdot \prod_{i=1}^{\infty} (1 - q^{i-1/2}x)^{-1} = q^{-D}(1 - q^{1/2}x)^{-1}, \\ & \prod_{i=1}^{\infty} (1 - q^{i-1/2}x) \cdot q^D \cdot \prod_{i=1}^{\infty} (1 - q^{i-1/2}x)^{-1} = (1 - q^{1/2}x)q^D, \end{aligned}$$

one obtains the result shown in (3.15). \square

Let us expand (3.15) and move $q^{\pm D}$ in each term to the right end. The outcome reads

$$A = (1 - q^{1/2}x)q^D + q^{1/2}x + Qq^{1/2}x + Qx^2(1 - q^{-1/2}x)^{-1}q^{-D}. \quad (3.16)$$

We are thus led to the following final expression of the quantum spectral curve of the melting crystal model.

Theorem 4. $Z(x)$ satisfies the equation

$$(A - 1)Z(x) = 0 \quad (3.17)$$

with respect to the q -difference operator (3.16).

4 Prescription for 4D limit

The 4D limit of the partition function $Z(\mathbf{0})$ at $\mathbf{t} = \mathbf{0}$ is achieved by setting the parameters as

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2 \quad (4.1)$$

and letting $R \rightarrow 0$ [17]. R is the radius of the fifth dimension of $\mathbb{R}^4 \times S^1$ in which SUSY Yang-Mills theory lives [18], \hbar is a parameter of the self-dual Ω background, and Λ is an energy scale of 4D $\mathcal{N} = 2$ SUSY Yang-Mills theory [19, 20]. The definition of 4D limit of $Z(x)$ and $Z(\mathbf{t})$ needs R -dependent transformations of x and \mathbf{t} .

4.1 4D limit of $Z(x)$ and quantum spectral curve

Alongside the substitution (4.1) of parameters, we transform the variable x to a new variable X as

$$x = x(X, R) = e^{R(X-\hbar/2)}. \quad (4.2)$$

As it turns out below, both the combinatorial expression (3.2) and the q -difference equation (3.17) of $Z(x)$ behave nicely as $R \rightarrow 0$ under this R -dependent transformation of x .

Lemma 6.

$$\lim_{R \rightarrow 0} Z(x(X, R)) = Z_{4D}(X),$$

where

$$Z_{4D}(X) = \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{i=1}^{\infty} \frac{X - (\lambda_i - i + 1)\hbar}{X - (-i + 1)\hbar}. \quad (4.3)$$

Proof. As $R \rightarrow 0$ under the R -dependent transformations (4.1) and (4.2), the building blocks of (3.2) behave as

$$\begin{aligned} s_{\lambda}(q^{-\rho})^2 &= \left(\frac{\dim \lambda}{|\lambda|} \right)^2 (R\hbar)^{-2|\lambda|} (1 + O(R)), \\ Q^{|\lambda|} &= (R\Lambda)^{2|\lambda|}, \\ \frac{1 - q^{\lambda_i - i + 1/2} x(X, R)}{1 - q^{-i + 1/2} x(X, R)} &= \frac{X - (\lambda_i - i + 1)\hbar}{X - (-i + 1)\hbar} (1 + O(R)). \end{aligned}$$

Note that the hook-length formulae (2.3) and (2.4) are used in the derivation of the first line above. \square

Lemma 7.

$$A - 1 = \left(-(X - \hbar)(e^{-\hbar d/dX} - 1) - \frac{\Lambda^2}{X} e^{\hbar d/dX} \right) R + O(R^2).$$

Proof. (3.16) implies that $A - 1$ can be expressed as

$$A - 1 = (1 - q^{1/2}x)(q^D - 1) + Qq^{1/2}x + Qx^2(1 - q^{-1/2}x)^{-1}q^{-D}.$$

As $R \rightarrow 0$ under the transformations (4.1) and (4.2), each term of this expression behaves as follows:

$$\begin{aligned} 1 - q^{1/2}x &= -R(X - \hbar) + O(R^2), \\ q^{\pm D} &= e^{\mp \hbar d/dX}, \\ Qq^{1/2}x &= O(R^2), \\ Qx^2(1 - q^{-1/2}x)^{-1}q^{-D} &= -\frac{\Lambda^2}{X}R + O(R^2). \end{aligned}$$

□

As a consequence of the foregoing two facts, we obtain the following difference equation for $Z_{4D}(X)$.

Theorem 5. $Z_{4D}(X)$ satisfies the difference equation

$$\left((X - \hbar)(e^{-\hbar d/dX} - 1) + \frac{\Lambda^2}{X} e^{\hbar d/dX} \right) Z_{4D}(X) = 0. \quad (4.4)$$

By the shift $X \rightarrow X + \hbar$ of X , (4.4) turns into the equation

$$\left(X(e^{-\hbar d/dX} - 1) + \frac{\Lambda^2}{X + \hbar} e^{\hbar d/dX} \right) Z_{4D}(X + \hbar) = 0,$$

which agrees with the equation derived by Dunin-Barkowski et al. [5]. Moreover, as they found, this equation can be converted to the simpler form

$$(e^{-\hbar d/dX} + \Lambda^2 e^{\hbar d/dX} - X) \Psi(X) = 0 \quad (4.5)$$

by the gauge transformation

$$\Psi(X) = \exp \left(B \left(-\hbar \frac{d}{dX} \right) \frac{X - X \log X}{\hbar} \right) Z_{4D}(X + \hbar),$$

where $B(t)$ is the generating function

$$B(t) = \frac{t}{e^t - 1}$$

of the Bernoulli numbers. It is this equation (4.5) that is identified by Dunin-Barkowski et al. [5] as the equation of quantum spectral curve for Gromov-Witten theory of \mathbb{CP}^1 . Its classical limit

$$y^{-1} + y - x = 0$$

as $\hbar \rightarrow 0$ (with Λ normalized to 1) is the spectral curve of topological recursion in this case [6, 7, 21]. We have thus rederived the quantum spectral curve of \mathbb{CP}^1 from the 4D limit of the melting crystal model.

4.2 4D limit of $Z(\mathbf{t})$

As shown in the proof of Lemma 6, the deformed Boltzmann weight $s_\lambda(q^{-\rho})^2 Q^{|\lambda|}$ behaves nicely in the limit as $R \rightarrow 0$. To achieve the 4D limit of $Z(\mathbf{t})$, we have only to find an appropriate R -dependent transformation $\mathbf{t} = \mathbf{t}(\mathbf{T}, R)$ to the coupling constants $\mathbf{T} = (T_1, T_2, \dots)$ of 4D external potentials $\phi_k^{4D}(\lambda)$ for which the identity

$$\lim_{R \rightarrow 0} \phi(\mathbf{t}(\mathbf{T}, R), \lambda) = \phi_{4D}(\mathbf{T}, \lambda) = \sum_{k=1}^{\infty} T_k \phi_k^{4D}(\lambda) \quad (4.6)$$

holds. In view of the definition (2.6) of $\phi_k(\lambda)$'s, it is natural to expect that $\phi_k^{4D}(\lambda)$'s take such a form as

$$\phi_k^{4D}(\lambda) = \sum_{i=1}^{\infty} ((\lambda_i - i + 1)^k - (-i + 1)^k). \quad (4.7)$$

The following is a clue to this problem.

Lemma 8. *As $R \rightarrow 0$ under the transformation (4.1) of the parameters,*

$$\sum_{j=1}^k (-1)^{k-j} \binom{k}{j} \phi_j(\lambda) = \phi_k^{4D}(\lambda) (-R\hbar)^k + O(R^{k+1}). \quad (4.8)$$

Proof. The difference of the two identities

$$\begin{aligned} \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} q^{ju} &= (q^u - 1)^k - (-1)^k, \\ \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} q^{jv} &= (q^v - 1)^k - (-1)^k \end{aligned}$$

yields the identity

$$\begin{aligned} \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} (q^{ju} - q^{jv}) &= (q^u - 1)^k - (q^v - 1)^k \\ &= (u^k - v^k)(-R\hbar)^k + O(R^{k+1}). \end{aligned}$$

One can derive (4.8) by specializing this identity to $u = \lambda_i - i + 1$ and $v = -i + 1$ and summing the outcome over $i = 1, 2, \dots$ \square

(4.8) implies the identity

$$\lim_{R \rightarrow 0} \sum_{k=1}^{\infty} \frac{T_k}{(-R\hbar)^k} \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} \phi_j(\lambda) = \sum_{k=1}^{\infty} T_k \phi_k^{4D}(\lambda)$$

for $\phi_k(\lambda)$'s and the potentials shown in (2.6). Since

$$\sum_{k=1}^{\infty} \frac{T_k}{(-R\hbar)^k} \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} \phi_j(\lambda) = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^{k-j} T_k}{(-R\hbar)^k} \phi_j(\lambda),$$

one can conclude that the identity (4.6) holds if t_k 's and T_k 's are related by the linear relations

$$t_j = \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^{k-j} T_k}{(-R\hbar)^k}. \quad (4.9)$$

This gives an R -dependent transformation $\mathbf{t} = \mathbf{t}(\mathbf{T}, R)$ that we have sought for. Note that this is a triangular (hence invertible) linear transformation between \mathbf{t} and \mathbf{T} .

Let $Z_{4D}(\mathbf{T})$ denote the deformed partition function

$$Z_{4D}(\mathbf{T}) = \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} e^{\phi_{4D}(\mathbf{T}, \lambda)} \quad (4.10)$$

with the external potentials (4.7). We are thus led to the following conclusion.

Theorem 6. *As $R \rightarrow 0$ under the R -dependent transformation $\mathbf{t} = \mathbf{t}(\mathbf{T}, R)$ of the coupling constants defined by (4.9), $Z(\mathbf{t})$ converges to $Z_{4D}(\mathbf{T})$:*

$$\lim_{R \rightarrow 0} Z(\mathbf{t}(\mathbf{T}, R)) = Z_{4D}(\mathbf{T}). \quad (4.11)$$

It is easy to see that $Z_{4D}(X)$ and $Z_{4D}(\mathbf{T})$ are connected by the substitution

$$T_k = -\frac{\hbar^k}{kX^k} \quad (4.12)$$

as

$$Z_{4D}(X) = Z_{4D} \left(-\frac{\hbar}{X}, -\frac{\hbar^2}{2X^2}, \dots, -\frac{\hbar^k}{kX^k}, \dots \right).$$

This fact plays a role in the next section.

5 Bilinear equations

5.1 Fay-type bilinear equations for KP hierarchy

Let us recall the notion of Fay-type bilinear equations in the theory of the KP hierarchy [1, 26, 33].

Given a general tau function $\tau(\mathbf{t})$, one can consider an N -variate generalization of (3.12):

$$\tau([x_1] + \dots + [x_N]) = \langle 0 | \Gamma_+(x_1, \dots, x_N) | W \rangle.$$

It product with the Vandermonde determinant

$$\Delta(x_1, \dots, x_N) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$$

is the N -point function of the fermion field $\psi^*(x^{-1})$ in the background state $|W\rangle$ [9, 16].

Actually, it is more convenient to leave \mathbf{t} as well. Let $\tau(\mathbf{t}, x_1, \dots, x_N)$ denote the function thus obtained:

$$\begin{aligned} \tau(\mathbf{t}, x_1, \dots, x_N) &= \tau(\mathbf{t} + [x_1] + \dots + [x_N]) \\ &= \langle 0 | \Gamma_+(x_1, \dots, x_N) \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) | W \rangle. \end{aligned} \quad (5.1)$$

By virtue of the aforementioned interpretation as the N -point function of a fermion field, the product

$$\xi(x_1, \dots, x_N) = \Delta(x_1, \dots, x_N) \tau(\mathbf{t}, x_1, \dots, x_N)$$

with the Vandermonde determinant satisfies the bilinear equations

$$\sum_{j=N}^{2N} (-1)^{j-N} \xi(x_1, \dots, x_{N-1}, x_j) \xi(x_N, \dots, \widehat{x_j}, \dots, x_{2N}) = 0, \quad (5.2)$$

where $\widehat{x_j}$ means removing x_j from the list of variables therein. As pointed out by Sato and Sato [26], these equations are avatars of the Plücker relations among the Plücker coordinates of a Grassmann manifold.

The simplest ($N = 2$) case

$$\begin{aligned} & (x_1 - x_2)(x_3 - x_4)\tau(\mathbf{t}, x_1, x_2)\tau(\mathbf{t}, x_3, x_4) \\ & - (x_1 - x_3)(x_2 - x_4)\tau(\mathbf{t}, x_1, x_3)\tau(\mathbf{t}, x_2, x_4) \\ & + (x_1 - x_4)(x_2 - x_3)\tau(\mathbf{t}, x_1, x_4)\tau(\mathbf{t}, x_2, x_3) = 0 \end{aligned} \quad (5.3)$$

of (5.2), referred to as a Fay-type bilinear equation, is known to play a particular role. Specialized to $x_4 = 0$, this equation turns into the so called Hirota-Miwa equation

$$\begin{aligned} & (x_1 - x_2)x_3\tau(\mathbf{t} + [x_1] + [x_2])\tau(\mathbf{t} + [x_3]) \\ & + (x_2 - x_3)x_1\tau(\mathbf{t} + [x_2] + [x_3])\tau(\mathbf{t} + [x_1]) \\ & + (x_3 - x_1)x_2\tau(\mathbf{t} + [x_3] + [x_1])\tau(\mathbf{t} + [x_2]) = 0. \end{aligned} \quad (5.4)$$

Moreover, dividing this equation by x_3 and letting $x_3 \rightarrow 0$ yield the differential Fay identity [1]

$$\begin{aligned} & (x_1 - x_2)(\tau(\mathbf{t} + [x_1] + [x_2])\tau(\mathbf{t}) - \tau(\mathbf{t} + [x_1])\tau(\mathbf{t} + [x_2])) \\ & + x_1x_2(\tau(\mathbf{t} + [x_1])\tau_{t_1}(\mathbf{t} + [x_2]) - \tau_{t_1}(\mathbf{t} + [x_1])\tau(\mathbf{t} + [x_2])) = 0, \end{aligned} \quad (5.5)$$

where $\tau_{t_1}(\mathbf{t})$ denotes the t_1 -derivative of $\tau(\mathbf{t})$. It is known [33] that the differential Fay identity characterizes a general tau function of the KP hierarchy in the following sense.

Theorem 7. *A function $\tau(\mathbf{t})$ of $\mathbf{t} = (t_1, t_2, \dots)$ is a tau function of the KP hierarchy if and only if it satisfies (5.5).*

As a corollary, it turns out that each of (5.3) and (5.4), too, is a necessary and sufficient condition for a function $\tau(\mathbf{t})$ to be a KP tau function. This fact is a clue to the subsequent consideration.

5.2 Bilinear equations in melting crystal model

Let $Z(\mathbf{t}, x_1, \dots, x_N)$ denote the function

$$Z(\mathbf{t}, x_1, \dots, x_N) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} e^{\phi(\mathbf{t}, \lambda)} \prod_{j=1}^N \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2} x_j}{1 - q^{-i + 1/2} x_j} \quad (5.6)$$

obtained by shifting \mathbf{t} in $Z(\mathbf{t})$ as

$$t_k \rightarrow t_k - \sum_{j=1}^N \frac{q^{-k/2} x_j^k}{k}.$$

This amounts to inserting an N -variate vertex operator in a fermionic expression of $Z(\mathbf{t})$, say (3.6), as

$$\begin{aligned} Z(\mathbf{t}, x_1, \dots, x_N) &= \langle 0 | \Gamma'_+(x_1, \dots, x_N) \exp \left(\sum_{k=1}^{\infty} (-1)^k q^{k/2} t_k J_k \right) g_2 | 0 \rangle \\ &= \langle 0 | \Gamma_+(x_1, \dots, x_N) \exp \left(- \sum_{k=1}^{\infty} q^{k/2} t_k J_k \right) g'_2 | 0 \rangle. \end{aligned} \quad (5.7)$$

Viewing (5.7) as a special case of (5.1), one can apply the aforementioned facts about KP tau functions. In particular, $Z(\mathbf{t})$ satisfies the three-term bilinear equation

$$\begin{aligned} &(x_1 - x_2)(x_3 - x_4)Z(\mathbf{t}, x_1, x_2)Z(\mathbf{t}, x_3, x_4) \\ &- (x_1 - x_3)(x_2 - x_4)Z(\mathbf{t}, x_1, x_3)Z(\mathbf{t}, x_2, x_4) \\ &+ (x_1 - x_4)(x_2 - x_3)Z(\mathbf{t}, x_1, x_4)Z(\mathbf{t}, x_2, x_3) = 0. \end{aligned} \quad (5.8)$$

It is remarkable that these bilinear equations survive the 4D limit. Let us set the parameters q, Q and the coupling constants \mathbf{t} to the R -dependent form shown in (4.1) and (4.9), and transform the variables x_1, \dots, x_N to new variables X_1, \dots, X_N as

$$x_j = e^{R(X_j - \hbar/2)}, \quad j = 1, \dots, N.$$

Note that we have slightly modified the relation (4.2) between x and X for convenience of the subsequent consideration. As $R \rightarrow 0$ under these R -dependent transformations, $Z(\mathbf{t}, x_1, \dots, x_N)$ converges to a function of the form

$$\begin{aligned} &Z_{4D}(\mathbf{T}, X_1, \dots, X_N) \\ &= \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} e^{\phi_{4D}(\mathbf{T}, \lambda)} \prod_{j=1}^N \prod_{i=1}^{\infty} \frac{X_j - (\lambda_i - i + 1)\hbar}{X_j - (-i + 1)\hbar}. \end{aligned} \quad (5.9)$$

Since the differences $x_i - x_j$ in $\Delta(x_1, \dots, x_N)$ behave as

$$x_i - x_j = R(X_i - X_j) + O(R^2),$$

the three-term bilinear equation (5.8), divided by R^2 before letting $R \rightarrow 0$, turns into the equation

$$\begin{aligned} &(X_1 - X_2)(X_3 - X_4)Z_{4D}(\mathbf{T}, X_1, X_2)Z_{4D}(\mathbf{T}, X_3, X_4) \\ &- (X_1 - X_3)(X_2 - X_4)Z_{4D}(\mathbf{T}, X_1, X_3)Z_{4D}(\mathbf{T}, X_2, X_4) \\ &+ (X_1 - X_4)(X_2 - X_3)Z_{4D}(\mathbf{T}, X_1, X_4)Z_{4D}(\mathbf{T}, X_2, X_3) = 0 \end{aligned} \quad (5.10)$$

for $Z_{4D}(\mathbf{T}, X_i, X_j)$'s. The more general bilinear equations (5.2), too, have 4D counterparts.

Let us note here that $Z_{4D}(\mathbf{T}, X_1, \dots, X_N)$ can be obtained from $Z_{4D}(\mathbf{T})$ by shifting \mathbf{T} as

$$T_k \rightarrow T_k - \sum_{j=1}^N \frac{\hbar^k}{k X_j^k}$$

just as $Z_{4D}(X)$ and $Z_{4D}(\mathbf{T})$ are connected by the substitution shown in (4.12). This is essentially the same relation as $\tau(\mathbf{t}, x_1, \dots, x_N)$ is obtained from $\tau(\mathbf{t})$ except that x_j 's are replaced by X_j^{-1} ³. Therefore, for comparison with the Fay-type bilinear equation (5.3) for KP tau functions, one should rewrite (5.10) as

$$\begin{aligned} & (X_1^{-1} - X_2^{-1})(X_3^{-1} - X_4^{-1})Z_{4D}(\mathbf{T}, X_1, X_2)Z_{4D}(\mathbf{T}, X_3, X_4) \\ & - (X_1^{-1} - X_3^{-1})(X_2^{-1} - X_4^{-1})Z_{4D}(\mathbf{T}, X_1, X_3)Z_{4D}(\mathbf{T}, X_2, X_4) \\ & + (X_1^{-1} - X_4^{-1})(X_2^{-1} - X_3^{-1})Z_{4D}(\mathbf{T}, X_1, X_4)Z_{4D}(\mathbf{T}, X_2, X_3) = 0. \end{aligned}$$

It is this equation that corresponds to (5.3) literally. According to Theorem 7, this equation is enough to deduce the following conclusion.

Theorem 8. $Z_{4D}(\mathbf{T})$ is a tau function of the KP hierarchy with time variables $\mathbf{T} = (T_1, T_2, \dots)$.

This result may be explained in the context of Gromov-Witten theory of \mathbb{CP}^1 as well. $Z_{4D}(\mathbf{T})$ has a fermionic expression, analogous to (2.11), of the form

$$Z_{4D}(\mathbf{T}) = \langle 0 | e^{J_1}(\Lambda/\hbar)^{2L_0} e^{H_{4D}(\mathbf{T})} e^{J_{-1}} | 0 \rangle, \quad (5.11)$$

where

$$H_{4D}(\mathbf{T}) = \sum_{k=1}^{\infty} T_k H_k^{4D}, \quad H_k^{4D} = \sum_{n \in \mathbb{Z}} n^k \cdot \psi_{-n} \psi_n^*.$$

This function is a member of the set of functions $Z_{4D}(\mathbf{T}, s)$, $s \in \mathbb{Z}$, defined as

$$Z_{4D}(\mathbf{T}, s) = \langle s | e^{J_1}(\Lambda/\hbar)^{2L_0} e^{H_{4D}(\mathbf{T})} e^{J_{-1}} | s \rangle,$$

where $|s\rangle$ and $\langle s|$ are the ground states of the charge- s sector of the Fock spaces. The combinatorial definition (4.10) of $Z_{4D}(\mathbf{T})$ can be extended to these functions as

$$Z_{4D}(\mathbf{T}, s) = \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda| + s(s+1)} e^{\phi_{4D}(\mathbf{T}, s, \lambda)}, \quad (5.12)$$

³The presence of the negative sign and the coefficients \hbar^k is not essential, because rescaling $t_k \rightarrow c^k t_k$ and reversal $t_k \rightarrow -t_k$ of the time variables are symmetries of the KP hierarchy. This is also the case for the relation between $Z(\mathbf{t}, x_1, \dots, x_N)$ and $Z(\mathbf{t})$.

where

$$\phi_{4D}(\mathbf{T}, s, \lambda) = \sum_{k=1}^{\infty} T_k \phi_k^{4D}(s, \lambda),$$

$$\phi_k^{4D}(s, \lambda) = \sum_{k=1}^{\infty} ((\lambda_i - i + 1 + s)^k - (-i + 1 + s)^k) + \text{correction terms}.$$

$Z_{4D}(\mathbf{T}, s)$ is also known as a generating function of all-genus Gromov-Witten invariants of \mathbb{CP}^1 [23], and proven to be a tau function of the 1D Toda hierarchy by several different methods [4, 8, 15]. One can deduce from these results, too, that $Z_{4D}(\mathbf{T}) = Z_{4D}(\mathbf{T}, 0)$ is a tau function of the KP hierarchy.

As far as the KP hierarchy is concerned, our proof is conceptually simpler than those in the aforementioned literature. Moreover, one can use a Toda version [29, 34] of Fay-type bilinear equations to prove in much the same way that $Z_{4D}(\mathbf{T}, s)$ is a tau function of the 1D Toda hierarchy, though the detail is slightly more complicated.

6 Conclusion

Primary motivation of this work was to understand the result of Dunin-Barkowski et al. [5] in the language of the quantum spectral curve of the melting crystal model. In the course of solving this problem, we have found how to achieve the 4D limit of the deformed partition function $Z(\mathbf{t})$ itself. As a byproduct, this prescription for 4D limit has turned out to transfer Fay-type bilinear equations from $Z(\mathbf{t})$ to its 4D limit $Z_{4D}(\mathbf{T})$.

It will be better to summarize these results from two aspects, namely, quantum curves and bilinear equations:

1. *Quantum curves*: One can derive a quantum spectral curve of the melting crystal model by the method of our work on quantum mirror curves in topological string theory [32]. This quantum curve is formulated as the q -difference equation (3.17) for the single-variate specialization $Z(x)$ of $Z(\mathbf{t})$. Its 4D limit is achieved by transforming the variable x to a new variable X as shown in (4.2) and letting $R \rightarrow 0$. (3.17) thereby turns into the difference equation (4.4) for the 4D version $Z_{4D}(X)$ of $Z(x)$. (4.4) can be further converted to the quantum spectral curve (4.5) of Gromov-Witten theory of \mathbb{CP}^1 . (4.4) and (4.5) are derived by Dunin-Barkowski et al. [5] by genuinely combinatorial computations. Our approach highlights a role of the KP hierarchy that underlies these quantum curves.

2. *Bilinear equations*: According to our previous work on the melting crystal model [17], $Z(\mathbf{t})$ is a tau function of the KP hierarchy. As $R \rightarrow 0$ under the R -dependent transformation (4.9) of the coupling constants, $Z(\mathbf{t})$ converges to the 4D version $Z_{4D}(\mathbf{T})$. In this limit, the three-term bilinear equation (5.8) for $Z(\mathbf{t})$ turns into its counterpart (5.10) for $Z_{4D}(\mathbf{T})$. This implies that $Z_{4D}(\mathbf{T})$, too, is a tau function of the KP hierarchy. We have thus obtained a new approach to the integrable structure in Okounkov and Pandharipande's generating function of all-genus Gromov-Witten invariants of \mathbb{CP}^1 [23].

Though the detail is omitted, our consideration on Fay-type bilinear equations carries over to the s -deformed partition functions $Z(\mathbf{t}, s)$ and $Z_{4D}(\mathbf{T}, s)$ defined by (2.7) and (5.12). This leads to yet another proof of the fact [4, 8, 15] that $Z_{4D}(\mathbf{T}, s)$ is a tau function of the 1D Toda hierarchy.

Let us stress that the integrable structure of $Z_{4D}(\mathbf{T})$ still remains to be fully elucidated. Its 5D (or K-theoretic) lift $Z(\mathbf{t})$ has a fermionic expression, such as (2.15) and (3.6), that shows manifestly that $Z(\mathbf{t})$ is a tau function of the KP hierarchy. Moreover, since the generating operator g therein is rather simple, one can even find the associated generating operator G in $V = \mathbb{C}((x))$ explicitly. In contrast, no similar fermionic expression of $Z_{4D}(\mathbf{T})$ is currently known. The preliminary fermionic expression (5.11) of $Z_{4D}(\mathbf{T})$ cannot be converted to such a form by the method of our previous work [17]. The limiting procedure from $Z(\mathbf{t})$ is a way to overcome this difficulty, but this should not be a final answer.

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